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Construction of the Geometry of Euclidean n-Dimensional Space by the Theory of Continuous Groups.

By E. O. Lovett.

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1. With regard to space, let it be assumed:

1°. That it is an n-dimensional manifoldness, i. e., that n independent data are necessary and sufficient to determine the position of an element of the manifoldness; these n independent things are called the coordinates of the element.

2°. That a figure of the manifoldness possesses $\frac{n(n+1)}{2}$ degrees of freedom within the manifoldness; i. e., that $\frac{n(n+1)}{2}$ independent data are necessary and sufficient to render a rigid body fixed in position; the latter $\frac{n(n+1)}{2}$ independent things are called the parameters of the figure.

For convenience, let the element be called a point, and its coordinates be designated by $x_1, x_2, \ldots, x_n$. Consider any figure containing this point and let the parameters of the figure be $\alpha_1, \alpha_2, \ldots, \alpha_{\frac{n(n+1)}{2}}$. Let the figure assume a new position and call $x'_1, x'_2, \ldots, x'_n$ the coordinates of the new position of $(x_1, x_2, \ldots, x_n)$. Then

$$x'_i = \xi_i(x_1, x_2, \ldots, x_n, \quad \alpha_1, \alpha_2, \ldots, \alpha_{\frac{n(n+1)}{2}}),$$
$$i = 1, 2, \ldots, n.$$

The operation changing $(x_1, x_2, \ldots, x_n)$ into $(\xi_1, \xi_2, \ldots, \xi_n)$ represents one of the motions of an n-dimensional figure, and the ensemble of all these operations constitutes a continuous group with $\frac{n(n+1)}{2}$ parameters. The identical transformation ought to appear among these operations; accordingly, there must be some system of parameters $\alpha_1, \ldots, \alpha_{\frac{n(n+1)}{2}}$, such that

$$\xi_1 = x_1, \quad \xi_2 = x_2, \ldots, \xi_n = x_n.$$

There is no loss of generality in assuming the preceding system of values to be

$$\alpha_1 = \alpha_2 = \ldots = \alpha_{\frac{n(n+1)}{2}} = 0.$$

An infinitesimal transformation of the group is one whose parameters differ by infinitesimals from those parameters which produce the identical transformation; in this case, the infinitesimal transformation is obtained by assigning infinitesimal values to the parameters themselves; i. e., by such a transformation $x_1, x_2, \ldots, x_n$ are changed respectively to

$$x_i + \sum_{1}^{\frac{n(n+1)}{2}}{}^{j} \alpha_j \frac{\partial \xi_i}{\partial \alpha_j}, \qquad i = 1, 2, \ldots, n,$$

in the partial derivatives of which the α's should be put equal to zero.

Writing, for short,

$$p_i = \frac{\partial f}{\partial x_i}, \qquad i = 1, 2, \ldots, n,$$

the common symbol for the above infinitesimal transformation is

$$I \equiv \sum_{1}^{n}{}^{i} p_i \sum_{1}^{\frac{n(n+1)}{2}}{}^{j} \alpha_j \frac{\partial \xi_i}{\partial \alpha_j},$$

or, putting

$$J_i \equiv \sum_{1}^{n}{}^{j} p_j \frac{\partial \xi_j}{\partial \alpha_i}, \qquad i = 1, 2, \ldots, \frac{n(n+1)}{2},$$

any infinitesimal transformation may be written

$$I \equiv \sum_{1}^{\frac{n(n+1)}{2}} \alpha_i J_i.$$

By a fundamental theorem of Lie, if we put

$$(J_i, J_j) \equiv \sum_{1}^{n}{}^{k} \left(\frac{\partial J_i}{\partial p_k} \frac{\partial J_j}{\partial x_k} - \frac{\partial J_i}{\partial x_k} \frac{\partial J_j}{\partial p_k} \right),$$

then

$$(J_i,\ J_j)=\sum_{1}^{\frac{n(n+1)}{2}}{}^{k}\ \lambda_k^{(i,\ j)}J_k, \tag{1}$$

$$i=1,\ 2,\ \ldots.,\ \frac{n(n+1)}{2},\quad j=1,\ 2,\ \ldots.,\ \frac{n(n+1)}{2},$$

where the λ's are constants. There are

$$\tfrac{1}{8}\left\{n^2(n+1)^2-2n(n+1)\right\}$$

of these equations, but the

$$\tfrac{1}{16}\left\{n^3(n+1)^3-2n^2(n+1)^2\right\}$$

λ's are not wholly arbitrary, since the following

$$\tfrac{1}{48}\left\{n^3(n+1)^3-6n^2(n+1)^2+8n(n+1)\right\}$$

identities of Jacobi

$$(J_i,\ (J_j,\ J_k))+(J_j,\ (J_k,\ J_i))+(J_k,\ (J_i,\ J_j))=0, \tag{2}$$

$$i, j,\ k=1,\ 2,\ \ldots.,\ \frac{n(n+1)}{2}$$

must hold.

Every set of J's satisfying (1) and (2) reveals a space whose independent infinitesimal motions are represented by the infinitesimal operators of the set. Those functions of the elements which are invariant under these transformations will characterize the geometry of the space. It is proposed here to find these characteristics for one set of operators.

2. The following forms for the fundamental transformations

$$J_1,\ J_2,\ \ldots.\ J_{\frac{n(n+1)}{2}}$$

$$p_1,\ p_2,\ \ldots.,\ p_n,\quad x_ip_j-x_jp_i, \tag{3}$$

$$i, j=C_{n,\,2}\ \text{of}\ 1,\ 2,\ \ldots.,\ n,$$

satisfy the conditions (1) and (2).

Let $(x_1,\ x_2,\ \ldots.,\ x_n)$ and $(x_1',\ x_2',\ \ldots.,\ x_n')$ be any two points, and $(\xi_1,\ \xi_2,\ \ldots.,\ \xi_n)$, $(\xi_1',\ \xi_2',\ \ldots.,\ \xi_n')$ be their positions after they are subjected to the transformation

$$I\equiv\sum_{1}^{\frac{n(n+1)}{2}}J_i,$$

where the J_i's have the values (3).

Let $\phi(x_1, \ldots, x_n, x'_1, \ldots, x'_n)$ be a function which is absolutely invariant under this operation; then, if such a function exist, we must have

$$\phi(\xi_1, \ldots, \xi_n, \xi'_1, \ldots, \xi'_n) = \phi(x_1, \ldots, x_n, x'_1, \ldots, x'_n),$$

that is,

$$I\phi \equiv 0$$

for all values of the α's; hence, equating to zero the coefficients of the α's in $I\phi$, we have the following system of linear partial differential equations for the function ϕ:

$$\left.\begin{aligned} &\frac{\partial\phi}{\partial x_i} + \frac{\partial\phi}{\partial x'_i} = 0, \qquad\qquad i = 1, 2, \ldots, n, \\ &x_i\frac{\partial\phi}{\partial x_j} + x'_i\frac{\partial\phi}{\partial x'_j} - x_j\frac{\partial\phi}{\partial x_i} - x'_j\frac{\partial\phi}{\partial x'_i} = 0, \\ &\qquad i, j = C_{n,2} \text{ of } 1, 2, \ldots, n. \end{aligned}\right\} \quad (4)$$

This system of $\frac{n(n+1)}{2}$ equations in $2n$ variables should not, in general, possess a solution. But the equations are not all independent. In fact, if we take the following system of $2n-1$ equations from the above, namely,

$$\frac{\partial\phi}{\partial x_\rho} + \frac{\partial\phi}{\partial x'_\rho} = 0, \qquad \rho = 1, 2, \ldots, n, \quad (5)$$

$$x_1\frac{\partial\phi}{\partial x_\sigma} - x_\sigma\frac{\partial\phi}{\partial x_1} + x'_1\frac{\partial\phi}{\partial x'_\sigma} - x'_\sigma\frac{\partial\phi}{\partial x'_1} = 0, \qquad \sigma = 2, 3, \ldots, n; \quad (6)$$

multiply the equations (5) respectively by

$$\frac{l_h^{(i,j)}}{x_1 - x'_1}, \qquad h = 1, 2, \ldots, n,$$

where

$$l_1^{(i,j)} = x_i x'_j - x_j x'_i, \quad l_k \quad = 0, \quad k = 2, \ldots, n;\ k \neq i \neq j$$
$$l_i^{(i,j)} = x_j x'_1 - x_1 x'_j, \quad l_j^{(i,j)} = x_1 x'_i - x_i x'_1;$$

multiply the equations (6) respectively by

$$\frac{\lambda_h^{(i,j)}}{x_1 - x'_1}, \qquad h = 2, 3, \ldots, n$$

where

$$\lambda_i^{(i,j)} = x'_j - x_j, \quad \lambda_j^{(i,j)} = x_i - x'_i,$$
$$\lambda_k^{(i,j)} = 0, \ k = 2, 3, \ldots, n;\ k \neq i \neq j;$$

and add the results, we obtain the remaining $\frac{1}{2}(n-1)(n-2)$ equations of the system (4), namely,

$$x_i\frac{\partial\phi}{\partial x_j}-x_j\frac{\partial\phi}{\partial x_i}+x'_i\frac{\partial\phi}{\partial x'_j}-x'_j\frac{\partial\phi}{\partial x'_i}=0.$$

The complete system (5) and (6) of $2n-1$ equations in $2n$ variables possesses at least one solution. That it has no more is readily seen by noting the fact that not all $(2n-1)^{\text{th}}$ order determinants of the matrix of $2n$ columns

$$\begin{Vmatrix}
0 & 0 & 0 & 0\ldots\ldots & 0 & 1 & 0 & 0 & 0\ldots\ldots & 0\\
0 & 0 & 0 & 0\ldots\ldots & 0 & 0 & 1 & 0 & 0\ldots\ldots & 0\\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\
0 & 0 & 0 & 0\ldots\ldots & 0 & 0 & 0 & 0 & 0\ldots\ldots & 1\\
x'_2-x_2 & x'_1-x_1 & 0 & 0\ldots\ldots & 0 & -x'_2 & x'_1 & 0 & 0\ldots\ldots & 0\\
x'_3-x_3 & 0 & x'_1-x_1 & 0\ldots\ldots & 0 & -x'_3 & 0 & x'_1 & 0\ldots\ldots & 0\\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\
x'_n-x_n & 0 & 0 & 0\ldots\ldots & x'_1-x_1 & -x'_n & 0 & 0 & 0\ldots\ldots & x'_1
\end{Vmatrix}$$

vanish; for example, the one formed by the last $2n-1$ columns whose value is $(x_1-x'_1)^{n-1}$.

The unique solution of this system appears by observing that the first n equations demand that the solution be a function of

$$X_i=x_i-x'_i,\qquad i=1,\,2,\,\ldots\ldots,\,n.$$

In these new variables the last $n-1$ equations become

$$X_i\frac{\partial\phi}{\partial X_j}-X_j\frac{\partial\phi}{\partial X_i}=0,\qquad i,j,=1,\,2,\,\ldots\ldots,\,n,$$

which require that ϕ be a function of

$$\sum_1^n X_i^2;$$

that is, the function

$$\delta=\sqrt{\sum_1^n(x_i-x'_i)^2} \tag{7}$$

is an absolute invariant under the most general tranformation of the group (3).

This function δ is said to define the distance between the two points $(x_1, x_2, \ldots, x_n)$, $(x_1', x_2', \ldots, x_n')$.

3. Consider, now, the linear manifoldness of elements

$$x_i + \lambda_i x_1 + \alpha_i = 0, \qquad i = 2, 3, \ldots, n.$$

The increments assigned to $x_1, x_2, \ldots, x_n$ by the transformations (3) are as follows:

$$\left.\begin{array}{l} \text{By } p_j \qquad \delta x_i = 1,\ i = j,\ \delta x_i = 0, i \neq j, \\ \qquad i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, n; \\ \text{By } x_i p_j - x_j p_i \quad \delta x_i = -x_j, \quad \delta x_j = +x_i, \quad \delta x_k = 0, \\ \qquad k \neq i \neq j = 1, 2, \ldots, n. \end{array}\right\} \tag{9}$$

By forming the variational equations

$$\delta(x_i + \lambda_i x_1 + \alpha_i) = 0$$

and substituting therein the values (9), we find the following values for the variations of λ_i and α_i under the transformations (3):

$$\left.\begin{array}{ll} \text{Under } p_i & \left\{\begin{array}{l} i = 1, \qquad \delta\lambda_j = 0, \quad \delta\alpha_j = -\lambda_j, \\ \qquad j = 2, 3, \ldots, n; \\ i = 2, 3, \ldots, n, \quad \delta\lambda_j = 0, \quad \delta\alpha_i = -1, \quad \delta\alpha_j = 0, \\ \qquad j = 2, 3, \ldots, n; \end{array}\right. \\ \text{Under } x_i p_j - x_j p_i & \left\{\begin{array}{l} i \neq 1, j \neq 1 \left\{\begin{array}{l} \delta\lambda_i = -\lambda_j, \quad \delta\lambda_j = \lambda_i, \quad \delta\alpha_i = -\alpha_j, \\ \qquad \delta\alpha_j = \alpha_i, \\ \delta\lambda_k = \delta\alpha_k = 0, \ k = 2, 3, \ldots, n \neq i \neq j; \end{array}\right. \\ i = 2, 3, \ldots, n,\ j = 1,\ \delta\lambda_i = \lambda_i^2 + 1,\ \delta\lambda_k = \lambda_i\lambda_k, \\ k = 2, 3, \ldots, n \neq i,\ \delta\alpha_{k'} = \alpha_i\lambda_{k'},\ k' = 2, 3, \ldots, n. \end{array}\right. \end{array}\right\} \tag{10}$$

The invariants of two linear manifoldnesses

$$(\lambda_2, \lambda_3, \ldots \lambda_n, \alpha_2, \alpha_3, \ldots, \alpha_n), \quad (\lambda_2', \lambda_3', \ldots \lambda_n', \alpha_2', \alpha_3', \ldots, \alpha_n')$$

are solutions of the system of partial differential equations

$$\overset{2}{\underset{n}{\sum}}{}^{i}\left\{\frac{\partial\phi}{\partial\lambda_i}\delta_j\lambda_i + \frac{\partial\phi}{\partial\alpha_i}\delta_j\alpha_i + \frac{\partial\phi}{\partial\lambda_i'}\delta_j\lambda_i' + \frac{\partial\phi}{\partial\alpha_i'}\delta_j\alpha_i\right\} = 0, \tag{11}$$

$$j = 1, 2, \ldots, \frac{n(n+1)}{2},$$

where the increments δ_j are to be successively replaced by those due to the $\frac{n(n+1)}{2}$ infinitesimal transformations (3) as tabulated above in (10).

For convenience, the equations (11) are arranged in the following manner:
$(n-1)$ equations:

$$\frac{\partial\phi}{\partial\alpha_i}+\frac{\partial\phi}{\partial\alpha_i'}=0, \qquad i=2,3,\ldots\ldots,n; \tag{12}$$

1 equation:

$$\sum_2^n\Big(\lambda_i\frac{\partial\phi}{\partial\alpha_i}+\lambda_i'\frac{\partial\phi}{\partial\alpha_i'}\Big)=0; \tag{13}$$

$(n-2)$ equations:

$$\lambda_2\frac{\partial\phi}{\partial\lambda_j}-\lambda_j\frac{\partial\phi}{\partial\lambda_2}+\alpha_2\frac{\partial\phi}{\partial\alpha_j}-\alpha_j\frac{\partial\phi}{\partial\alpha_2}+\lambda_2'\frac{\partial\phi}{\partial\lambda_j'}-\lambda_j'\frac{\partial\phi}{\partial\lambda_2'}$$
$$+\alpha_2'\frac{\partial\phi}{\partial\alpha_j'}-\alpha_j'\frac{\partial\phi}{\partial\alpha_2'}=0, \qquad j=3,4,\ldots\ldots,n; \tag{14}$$

$\frac{1}{2}(n-2)(n-3)$ equations:

$$\lambda_i\frac{\partial\phi}{\partial\lambda_j}-\lambda_j\frac{\partial\phi}{\partial\lambda_i}+\alpha_i\frac{\partial\phi}{\partial\alpha_j}-\alpha_j\frac{\partial\phi}{\partial\alpha_i}+\lambda_i'\frac{\partial\phi}{\partial\lambda_j'}-\lambda_j'\frac{\partial\phi}{\partial\lambda_i'}$$
$$+\alpha_i'\frac{\partial\phi}{\partial\alpha_j'}-\alpha_j'\frac{\partial\phi}{\partial\alpha_i'}=0, \qquad i=3,4,\ldots\ldots,n,\ j=2,3,\ldots\ldots,n; \tag{15}$$

$(n-1)$ equations:

$$(\lambda_j^2+1)\frac{\partial\phi}{\partial\lambda_j}+(\lambda_j'^2+1)\frac{\partial\phi}{\partial\lambda_j'}+\sum_{i=2\neq j}^{i=n\neq j}\Big(\lambda_j\lambda_i\frac{\partial\phi}{\partial\lambda_i}+\lambda_j'\lambda_i'\frac{\partial\phi}{\partial\lambda_i'}\Big)$$
$$+\sum_{i=2}^{i=n}\alpha_j\alpha_i\frac{\partial\phi}{\partial\alpha_i}+\alpha_j'\alpha_i'\frac{\partial\phi}{\partial\alpha_i'}\Big)=0 \qquad j=2,3,\ldots\ldots,n. \tag{16}$$

It will be seen from table (10) that the variations of the λ's are wholly independent of the α's; accordingly, invariant functions of the λ's and λ''s alone may exist. If such invariant functions $\psi(\lambda_2,\ldots\ldots,\lambda_n,\lambda_2',\ldots\ldots,\lambda_n')$ do exist they are solutions of the system of $\frac{n(n-1)}{2}$ equations in $2(n-1)$ variables:

$\frac{(n-2)(n-3)}{2}$ equations:

$$\lambda_i\frac{\partial\psi}{\partial\lambda_j}-\lambda_j\frac{\partial\psi}{\partial\lambda_i}+\lambda_i'\frac{\partial\psi}{\partial\lambda_j'}-\lambda_j'\frac{\partial\psi}{\partial\lambda_i'}=0, \tag{17}$$
$$i=3,4,\ldots\ldots,n,\ j=2,3,\ldots\ldots,n;$$

$(n-2)$ equations:

$$\lambda_2 \frac{\partial \psi}{\partial \lambda_j} - \lambda_j \frac{\partial \psi}{\partial \lambda_2} + \lambda_2' \frac{\partial \psi}{\partial \lambda_j'} - \lambda_j' \frac{\partial \psi}{\partial \lambda_2'} = 0, \qquad j = 3, 4, \ldots\ldots, n; \tag{18}$$

$(n-1)$ equations:

$$(\lambda_j^2+1) \frac{\partial \psi}{\partial \lambda_j} + (\lambda_j'^2+1) \frac{\partial \psi}{\partial \lambda_j'} + \sum_{i=2 \neq j}^{i=n \neq j} \left(\lambda_j \lambda_i \frac{\partial \psi}{\partial \lambda_i} + \lambda_j' \lambda_i' \frac{\partial \psi}{\partial \lambda_i'} \right) = 0, \qquad j = 2, 3, , \ldots\ldots, n. \tag{19}$$

This system cannot possess a solution if the equations are all independent. That the number of independent equations in the system is not greater than $2n-3$ we verify by observing that the $\frac{(n-2)(n-3)}{2}$ determinants of the $2(n-1)^{\text{th}}$ order formed by appending the rows of the array:

$$\left\{\begin{array}{cccccccccccccc}
0 & -\lambda_n & 0 & 0 \ldots\ldots & 0 & 0 & \lambda_3 & 0 & -\lambda_n' & 0 & 0 \ldots\ldots & 0 & 0 & \lambda_3' \\
0 & 0 & -\lambda_n & 0 \ldots\ldots & 0 & 0 & \lambda_4 & 0 & 0 & -\lambda_n' & 0 \ldots\ldots & 0 & 0 & \lambda_4' \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 \ldots\ldots & 0 & -\lambda_n & \lambda_{n-1} & 0 & 0 & 0 & 0 \ldots\ldots & 0 & -\lambda_n' & \lambda_{n-1}' \\
0 & -\lambda_{n-1} & 0 & 0 \ldots\ldots & 0 & \lambda_3 & 0 & 0 & -\lambda_{n-1}' & 0 & 0 \ldots\ldots & 0 & \lambda_3' & 0 \\
0 & 0 & -\lambda_{n-1} & 0 \ldots\ldots & 0 & \lambda_4 & 0 & 0 & 0 & -\lambda_{n-1}' & 0 \ldots\ldots & 0 & \lambda_4' & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 \ldots\ldots & -\lambda_{n-1} & \lambda_{n-2} & 0 & 0 & 0 & 0 & 0 \ldots\ldots & -\lambda_{n-1}' & \lambda_{n-2}' & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & -\lambda_4 & \lambda_3 & 0 \ldots\ldots & 0 & 0 & 0 & 0 & -\lambda_4' & \lambda_3' & 0 \ldots\ldots & 0 & 0 & 0
\end{array}\right\} \tag{20}$$

successively to the matrix

$$\left\| \begin{array}{cccccccccccc}
-\lambda_3 & \lambda_2 & 0 & 0 & \ldots\ldots & 0 & -\lambda_3' & \lambda_2' & 0 & 0 & \ldots\ldots & 0 \\
-\lambda_4 & 0 & \lambda_2 & 0 & \ldots\ldots & 0 & -\lambda_4' & 0 & \lambda_2' & 0 & \ldots\ldots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
-\lambda_n & 0 & 0 & 0 & \ldots\ldots & \lambda_2 & -\lambda_n' & 0 & 0 & 0 & \ldots\ldots & \lambda_2' \\
\lambda_2^2+1 & \lambda_2\lambda_3 & \lambda_2\lambda_4 & \lambda_2\lambda_5 & \ldots\ldots & \lambda_2\lambda_n & \lambda_2'^2+1 & \lambda_2'\lambda_3' & \lambda_2'\lambda_4' & \lambda_2'\lambda_5' & \ldots\ldots & \lambda_2'\lambda_n' \\
\lambda_2\lambda_3 & \lambda_3^2+1 & \lambda_3\lambda_4 & \lambda_3\lambda_5 & \ldots\ldots & \lambda_3\lambda_n & \lambda_2'\lambda_3' & \lambda_3'^2+1 & \lambda_3'\lambda_4' & \lambda_3'\lambda_5' & \ldots\ldots & \lambda_3'\lambda_n' \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\lambda_2\lambda_n & \lambda_3\lambda_n & \lambda_4\lambda_n & \lambda_5\lambda_n & \ldots\ldots & \lambda_n^2+1 & \lambda_2'\lambda_n' & \lambda_3'\lambda_n' & \lambda_4'\lambda_n' & \lambda_5'\lambda_n' & \ldots\ldots & \lambda_n'^2+1
\end{array} \right\| \tag{21}$$

are equal to zero.

Consider, then, the system formed by the $(2n-3)$ equations (18) and (19) in the $2(n-1)$ variables $\lambda_2, \ldots\ldots, \lambda_n, \lambda_2', \ldots\ldots, \lambda_n$.

The equations (18) assert that ψ is a function of

$$\xi \equiv \sum_2^n \lambda_i^2, \quad \eta \equiv \sum_2^n \lambda_i'^2, \quad \zeta \equiv \sum_2^n \lambda_i \lambda_i'. \tag{22}$$

The equations (19) become in the new variables ξ, η, ζ:

$$2\lambda_j \xi_1 \frac{\partial \phi}{\partial \xi_1} + 2\lambda_j' \eta_1 \frac{\partial \phi}{\partial \eta_1} + (\lambda_j + \lambda_j') \zeta_1 \frac{\partial \phi}{\partial \zeta_1} = 0, \tag{23}$$

$$j = 2, 3, \ldots, n,$$

where $\xi_1 = \xi + 1, \quad \eta_1 = \eta + 1, \quad \zeta_1 = \zeta + 1.$

From any one of these last $(n - 1)$ equations we have

$$\frac{d\xi_1}{\xi_1} + \frac{d\eta_1}{\eta_1} - 2\frac{d\zeta_1}{\zeta_1} = 0,$$

that is, ϕ is a function of

$$\zeta_1^2 / \xi_1 \eta_1,$$

or the quantity

$$\cos^2 \theta = \frac{\left\{\sum_2^n (\lambda_i \lambda_i' + 1)\right\}^2}{\left\{\sum_2^n \lambda_i^2 + 1\right\}\left\{\sum_2^n \lambda_i'^2 + 1\right\}} \tag{24}$$

is an absolute invariant under the most general transformation of the group (3). The angle θ is said to be the angle between the two linear manifoldnesses.

That the above solution is unique is seen from the fact that not every $2(n - 1)^{\text{th}}$ order determinant of the matrix (21) vanishes.

As appears in the table (10), the variations of the α's are functions of the α's and λ's; we should not expect, then, to find an invariant function of the α's alone.

But the system composed of the equations (12), (13), (14), (15) and (16) yields $(n - 1)$ invariants, functions of the α's and λ's, which come to light in the following manner:

The equations (12) assert that ϕ is a function of

$$\rho_i \equiv \alpha_i - \alpha_i', \qquad i = 2, 3, \ldots, n.$$

In the latter variables the equation (13) becomes

$$\sum_2^n \sigma_j \frac{\partial \phi}{\partial \rho_j} = 0, \quad \sigma_j = \lambda_j - \lambda_j'. \qquad j = 2, 3, \ldots, n,$$

which requires that ϕ be a function of any set of the $(n-1)$ sets of $(n-2)$ determinants each,

$$\begin{array}{ll} \rho_2\sigma_j-\rho_j\sigma_2\equiv(\rho\sigma)_{2j}, & j=2,3,\ldots\ldots,n; \\ \rho_3\sigma_j-\rho_j\sigma_3\equiv(\rho\sigma)_{3j}, & j=2,3,\ldots\ldots,n; \\ \ldots\ldots\ldots\ldots\ldots\ldots, & \\ \rho_n\sigma_j-\rho_j\sigma_n\equiv(\rho\sigma)_{nj}, & j=2,3,\ldots\ldots,n. \end{array}$$

The $\frac{(n-1)(n-2)}{2}$ equations (14) and (15) become the $(n-1)$ following:

$$(\rho\sigma)_{ki}\frac{\partial\phi}{\partial(\rho\sigma)_{kj}}-(\rho\sigma)_{kj}\frac{\partial\phi}{\partial(\rho\sigma)_{ki}}=0, \qquad ij=2,3,\ldots\ldots,n,$$

which demand that ϕ be a function of

$$P\equiv\sum_{i=2}^{i=n}(\rho\sigma)_{ki}^2;$$

furthermore, in the original variables, the equations (14) and (15) assert that the λ's and λ''s enter into ϕ alone only by means of the determinants

$$\left\|\begin{array}{cccccc} \lambda_2 & \lambda_3 & \lambda_4 & \ldots\ldots & \lambda_n & 1 \\ \lambda_2' & \lambda_3' & \lambda_4' & \ldots\ldots & \lambda_n' & 1 \end{array}\right\|$$

and, in fact, only through the forms

$$Q\equiv\left\|\begin{array}{ccccc} \lambda_2 & \lambda_3 & \lambda_4 & \ldots\ldots & \lambda_n \\ \lambda_2' & \lambda_3' & \lambda_4' & \ldots\ldots & \lambda_n \end{array}\right\|^2,$$

and

$$R\equiv\sum_2^n(\lambda_i-\lambda_i')^2;$$

further, the equations (16) cannot be satisfied unless R and Q enter into the combination

$$Q+R\equiv\left\|\begin{array}{cccccc} \lambda_2 & \lambda_3 & \lambda_4 & \ldots\ldots & \lambda_n & 1 \\ \lambda_2' & \lambda_3' & \lambda_4' & \ldots\ldots & \lambda_n' & 1 \end{array}\right\|^2\equiv T.$$

Finally, in the variables P and T the last $(n-1)$ equations assume the form

$$(\lambda_j+\lambda_j')\left\{P\frac{\partial\phi}{\partial P}+T\frac{\partial\phi}{\partial T}\right\}=0, \qquad j=2,3,\ldots\ldots,n,$$

that is, the solution ϕ is an arbitrary function of

$$P/T.$$

Accordingly, the following $(n-1)$ expressions

$$\Delta_j^2 \equiv \frac{\sum\limits_{k=2}^{k=n} \begin{vmatrix} \lambda_j - \lambda_j' & \lambda_k - \lambda_k' \\ \alpha_j - \alpha_j' & \alpha_k - \alpha_k' \end{vmatrix}^2}{\begin{Vmatrix} \lambda_2 & \lambda_3 & \lambda_4 & \ldots\ldots & \lambda_n & 1 \\ \lambda_2' & \lambda_3' & \lambda_4' & \ldots\ldots & \lambda_n' & 1 \end{Vmatrix}^2}, \qquad j = 2, 3, \ldots\ldots n,$$

are invariant functions of the parameters of the two linear manifoldnesses under the most general transformation of the group (3).

When the two linear manifoldnesses intersect, all the Δ_j's are zero; conversely, when any Δ_j is zero, all the others are zero, and the linear manifoldnesses have a point in common.

It is clear, *a fortiori*, that the form

$$\Delta^2 \equiv \frac{\begin{Vmatrix} \lambda_2 - \lambda_2' & \lambda_3 - \lambda_3' & \ldots\ldots & \lambda_n' - \lambda_n' \\ \alpha_2 - \alpha_2' & \alpha_3 - \alpha_3' & \ldots\ldots & \alpha_n - \alpha_n' \end{Vmatrix}^2}{\begin{Vmatrix} \lambda_2 & \lambda_3 & \lambda_4 & \ldots\ldots & \lambda_n & 1 \\ \lambda_2' & \lambda_3' & \lambda_4' & \ldots\ldots & \lambda_n' & 1 \end{Vmatrix}^2}$$

is an invariant. The analogue of the forms Δ_j and Δ to the expression for the distance between the straight lines in ordinary space is apparent.

The fundamental notions of distance and direction in space of n-dimensions are thus introduced by the invariants (7) and (24), the former relative to two elements of the space, the latter relative to two simplest manifoldnesses composed of a simply infinite number of these elements. All the derived notions of geometry may then be derived by as simple extensions of these primary notions successively from three dimensions as occur when passing from the plane to ordinary space.

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